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On the central limit theorem and law of the iterated logarithm for stationary processes with applications to linear processes

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Abstract

Many of the proofs of various central limit theorems and laws of the iterated logarithm for strictly stationary processes are based on approximating martingales. Here we study on this line the functional central limit theorem and law of the iterated logarithm for stationary processes, not necessarily possessing a coboundary decomposition, with applications to stationary linear processes.

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1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space with an ergodic one-to-one bimeasurable measure preserving transformation T, and let $\{X_k = X_0 \circ T^k, -\infty < k < \infty\}$ be a (strictly) stationary ergodic sequence with $X_0 \in L^2(P)$. One possible method to obtain ordinary and functional central limit theorems (CLTs) and laws of the iterated logarithm (LILs) for $\{S_n = \sum_{k=1}^n X_k, n \geqslant 1\}$ is to approximate this sequence sufficiently closely by a martingale. For this, let $\mathscr{F}_0$ be a sub-σ-field of $\mathscr{F}$ such that $\mathscr{F}_0 \subset T^{-1}\mathscr{F}_0$, and set $\mathscr{F}_k = T^{-k}\mathscr{F}_0$ for every integer k. Let H_k denote the Hilbert subspace of $L^2(P)$ of all random variables measurable w.r.t. $\mathscr{F}_k$. If $Y_0 \in H_0 \ominus H_{-1}$, then $Y_k = Y_0 \circ T^k \in H_k \ominus H_{k-1}$ for every k, and hence $E(Y_k \mid \mathscr{F}_{k-1}) = 0$ a.s. Therefore $\{Y_k, \mathscr{F}_k, -\infty < k < \infty\}$ is a square integrable, stationary ergodic martingale difference sequence, and thus if $\sigma^2 = EY_0^2 > 0$, then ordinary and functional CLTs and LILs are available for $\{T_n = \sum_{k=1}^n Y_k, n \geqslant 1\}$. Consequently, if for a given sequence $\{X_k\}$ a martingale difference sequence $\{Y_k\}$ can be found such that $U_n = S_n - T_n$ is sufficiently small, then ordinary and functional CLTs and LILs carry over from $\{T_n\}$

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to $\{S_n\}$. For example, if

$$\lim_{n\to\infty} n^{-1} EU_n^2 = 0 \tag{1}$$

or (1) and (with $\log_2 n = \log\log n$)

$$\lim_{n\to\infty} U_n/(n\log_2 n)^{1/2} = 0 \quad \text{a.s.} \tag{2}$$

hold, then the CLT and LIL hold for S_n, respectively. In fact, since $ET_n^2 = n\sigma^2$, we obtain for $\sigma_n^2 = ES_n^2$ that

$$\begin{aligned}|\sigma_n^2 - n\sigma^2| &\leqslant 2|E(T_nU_n)| + EU_n^2 \leqslant 2(ET_n^2)^{1/2}(EU_n^2)^{1/2} + EU_n^2 \\ &= 2n^{1/2}\sigma(EU_n^2)^{1/2} + EU_n^2\end{aligned}$$

so that $n^{-1}\sigma_n^2 \to \sigma^2$, if (1) holds. Hence by the theorem in Billingsley (1961), see also Volný (1993, Theorem 1), $\sigma_n^{-1}S_n$ converges in distribution to $N(0,1)$. On the other hand, by the theorem in Stout (1970), we have

$$\limsup_{n\to\infty} T_n/(2n\sigma^2\log_2 n)^{1/2} = 1 \quad \text{a.s.}$$

so that

$$\limsup_{n\to\infty} S_n/(2\sigma_n^2\log_2 n)^{1/2} = 1 \quad \text{a.s.}$$

if (1) and (2) hold.

A well-known sufficient condition for (1) and (2) to hold is the existence of a coboundary decomposition for X_0, that is, the existence of a random variable $Y_0 \in H_0 \ominus H_{-1}$ with $\sigma^2 > 0$ and of a random variable $Z_0 \in L^2(P)$ such that

$$X_0 = Y_0 + Z_0\circ T - Z_0, \tag{3}$$

since then

$$U_n = S_n - T_n = Z_0\circ T^{n+1} - Z_0\circ T \quad \text{for all } n \geqslant 1,$$

from which (1) and (2) are immediate. This method for constructing the approximating martingale difference sequence $\{Y_k\}$ is due to Gordin (1969) and has been further developed by several authors (see e.g. Hall and Heyde, 1980; Heyde, 1975; Heyde and Scott, 1973; Volný, 1993). Among other things it has been shown that (3) implies not only (1) and (2), but also functional forms of the CLT and LIL.

In the present paper, we study the functional CLT and LIL for stationary processes $\{X_k\}$ not necessarily satisfying (3). We only require the existence of *some* stationary and ergodic approximating martingale difference sequence $\{Y_k\}$ and give conditions under which $U_n = S_n - T_n$ is sufficiently small for the approximation approach to work. More precisely, it will be shown in Proposition 2 below that an appropriate moment bound on $|U_n|^p$ for some $p > 2$ suffices to obtain a functional CLT and LIL for $\{X_k\}$, whatever the approximating martingale difference sequence $\{Y_k\}$ may be.

This result will then be applied to stationary linear processes. It turns out that our approach can cover cases which are not covered by the known results based on (3).

2. CLTs and LILs for stationary processes from approximating martingales via moment conditions

As announced in the introduction, let $\{X_k,\ -\infty < k < \infty\}$ be a (strictly) stationary ergodic sequence with $X_0 \in L^2(P)$, and let $\{Y_k,\ \mathscr{F}_k,\ -\infty < k < \infty\}$ be a square integrable stationary ergodic martingale difference sequence, both defined on $(\Omega, \mathscr{F}, P)$, but without any a priori relationship. Also, as before, set $S_n = \sum_{k=1}^n X_k$, $T_n = \sum_{k=1}^n Y_k$ and $U_n = S_n - T_n$ for every $n \geqslant 1$.

Let $g(n)$ be a positive function satisfying

$$\liminf_{n\to\infty} g(An)/g(n) > A \quad \text{for some integer } A \geqslant 2. \tag{4}$$

and for each $\varepsilon > 0$, there exists $\rho = \rho(\varepsilon) < 1$ such that

$$\limsup_{n\to\infty} \left\{ \max_{\rho n \leqslant i \leqslant n} g(i)/g(n) \right\} < 1 + \varepsilon. \tag{5}$$

Let $\log_m x = \log(\log_{m-1} x)$, and for simplicity of notation $\log_m x = 1$ for $0 < x \leqslant y$ with y being defined by $\log_m y = 1$. Lai and Stout (1980) proved strong limit theorems involving the following proposition.

Proposition 1. *Let $p > 0$. If*

$$E|U_n|^p \leqslant g(n), \quad n \geqslant 1, \tag{6}$$

then for every $\delta > 0$ and $m = 1, 2, \ldots,$

$$\lim_{n\to\infty} U_n/\{g(n)(\log n)\cdots(\log_m n)^{1+\delta}\}^{1/p} = 0 \quad \text{a.s.} \tag{7}$$

For $p > 2$, suppose

$$g(n) \sim C(n\log_2 n)^{p/2}\{(\log n)\cdots(\log_m n)^{1+\delta}\}^{-1} \tag{8}$$

as $n \to \infty$ for some $\delta > 0$, $m \geqslant 1$ and positive constant C. Since $p > 2$, (4) and (5) are satisfied. If the moment condition (6) holds with $g(n)$ in (8), then both the CLT and LIL hold for S_n. In fact,

$$EU_n^2 \leqslant (E|U_n|^p)^{2/p} \leqslant \{g(n)\}^{2/p},$$

and since $n^{-1}\{g(n)\}^{2/p} \to 0$, (1) follows. (2) is immediate from (7). Further the functional CLT and LIL also hold (Proposition 2).

Define sequences of random functions $\{\theta_n(\cdot)\}$ and $\{\eta_n(\cdot)\}$ on [0, 1], respectively, by

$$\theta_n(0) = 0, \quad \theta_n(t) = \sigma_n^{-1}\{S_k + (nt - k)X_{k+1}\}, \quad k \leqslant nt \leqslant k+1, \quad k = 0, 1, \dots, n-1 \tag{9}$$

and

$$\eta_n(0) = 0, \quad \eta_n(t) = (2\sigma_n^2 \log_2 \sigma_n^2)^{-1/2}\{S_k + (nt - k)X_{k+1}\},$$
$$k \leqslant nt \leqslant k+1, \quad k = 0, 1, \dots, n-1. \tag{10}$$

Let W be a standard Brownian motion on [0,1]. Let $C[0,1]$ be the space of all real-valued continuous functions on [0,1] endowed with the sup norm, and let $K \subset C[0,1]$ be the set of absolutely continuous functions (with respect to Lebesgue measure) such that

$$f(0) = 0 \quad \text{and} \quad \int_0^1 \{f'(x)\}^2 \, \mathrm{d}x \leqslant 1.$$

Proposition 2. *Let $p > 2$. If*

$$E|U_n|^p \leqslant C(n \log_2 n)^{p/2} \{(\log n) \cdots (\log_m n)^{1+\delta}\}^{-1},$$

for all $n \geqslant 1$ and some $\delta > 0$, $m \geqslant 1$ and $C < \infty$, then $\{\theta_n\}$ converges weakly in $C[0,1]$ to W. Also, $\{\eta_n\}$ is relatively compact in $C[0,1]$ with probability one, and the set of its limit points coincides with K.

Proof. Put $L(n) = (\log_2 n)\{(\log n) \cdots (\log_m n)^{1+\delta}\}^{-2/p}$. Let

$$M_n = \max_{1 \leqslant k \leqslant n} |U_k|.$$

Since $\{X_k - Y_k\}$ is stationary, from Theorem 5 in Lai and Stout (1980), generalizing Serfling's maximal inequality, there exists a constant C_1 such that

$$EM_n^p \leqslant C_1 \{nL(n)\}^{p/2}, \quad n \geqslant 1.$$

Since $L(n) \to 0$, by Chebyshev's inequality,

$$\lim_{n \to \infty} n^{-1/2} M_n = 0 \quad \text{in probability.}$$

On the other hand, by the Borel–Cantelli lemma,

$$\lim_{j \to \infty} M_{2^j}/(2^j \log_2 2^j)^{1/2} = 0 \quad \text{a.s.} \tag{11}$$

For $2^j \leqslant n \leqslant 2^{j+1}$,

$$M_n/(n \log_2 n)^{1/2} \leqslant M_{2^{j+1}}/(2^j \log_2 2^j)^{1/2}. \tag{12}$$

From (11), the right-hand side of (12) approaches zero almost surely as $j \to \infty$, and therefore

$$\lim_{n \to \infty} M_n/(n \log_2 n)^{1/2} = 0 \quad \text{a.s.}$$

Since $n^{-1}\sigma_n^2 \to \sigma^2$, the rest of the proof is the same as that of Theorem 5.5 in Hall and Heyde (1980).

3. Application to stationary linear processes

We may now consider the application of Proposition 2 to stationary linear processes. Under the same situation as in Section 1, let $\varepsilon_0 \in H_0 \ominus H_{-1}$, $\varepsilon_k = \varepsilon_0 \circ T^k$ $(-\infty < k < \infty)$ and $\sigma^2 = E\varepsilon_0^2 > 0$. Define a stationary ergodic linear process $\{X_k, -\infty < k < \infty\}$ by

$$X_k = \sum_{j=-\infty}^{\infty} \alpha_j \varepsilon_{k-j}, \quad \sum_{j=-\infty}^{\infty} \alpha_j^2 < \infty. \tag{13}$$

Then its spectral density is

$$f(\lambda) = (2\pi)^{-1}\sigma^2 \left| \sum_{j=-\infty}^{\infty} \alpha_j \mathrm{e}^{i\lambda j} \right|^2, \quad -\pi \leqslant \lambda \leqslant \pi.$$

Assume further that $0 < |\sum_{j=-\infty}^{\infty} \alpha_j| < \infty$. Put

$$Y_k = \left(\sum_{j=-\infty}^{\infty} \alpha_j \right) \varepsilon_k \quad \text{and} \quad T_n = \sum_{k=1}^{n} Y_k.$$

Then $\{Y_k\}$ is a stationary ergodic martingale difference sequence. If

$$\sum_{k=1}^{\infty} \left\{ \left(\sum_{j=k}^{\infty} \alpha_j \right)^2 + \left(\sum_{j=k}^{\infty} \alpha_{-j} \right)^2 \right\} < \infty, \tag{14}$$

then $X_0 - Y_0$ is a coboundary (see Hall and Heyde, 1980; Heyde, 1975; Volný, 1993). Hence condition (14) implies the functional CLT and LIL for S_n. Condition (14) covers the case where the α_n vary continually in sign. For example, $\alpha_{|n|} = (-1)^n n^{-1}$, $n \geqslant 1$. If the α_n are ultimately all positive, (14) covers the case $\alpha_{|n|} \sim Cn^{-3/2}(\log n)^{-1}$ as $n \to \infty$ for some $C > 0$, but not the cases $\alpha_{|n|} \sim Cn^{-\lambda}$ for some $1 < \lambda \leqslant 3/2$, and $\alpha_{|n|} \sim Cn^{-1}(\log n)^{-1-\varepsilon}$ for some $\varepsilon > 0$. We shall show a result for the functional CLT and LIL which covers the latter cases.

Lemma 1. *Suppose that*

$$|\alpha_n| = \mathrm{O}(|L'(|n|)|) \tag{15}$$

for some nonincreasing slowly varying function $L(x)$, with nondecreasing derivative $L'(x)$. Then there exist constants K_1 and K_2 such that

$$\sum_{|j| \geqslant n} |\alpha_j| \leqslant K_1 L(n), \quad n \geqslant 1, \tag{16}$$

and

$$\sum_{|j| \leqslant n} |j||\alpha_j| \leqslant K_2 n L(n), \quad n \geqslant 1. \tag{17}$$

Proof. Since we will only be concerned with large values of x, without loss of generality we assume that

$$L(x) = M \exp\left\{\int_B^X \frac{\varepsilon(t)}{t} \mathrm{d}t\right\}, \quad 0 < B < 1,$$

where $M > 0$ is a constant, and $\varepsilon(t)$ is continuous, $\varepsilon(t) \to 0$ as $t \to \infty$ and $\varepsilon(t) \leqslant 0$ for all $t \geqslant B$ (see Seneta, 1976). Put $h(x) = -x^{-1}\varepsilon(x)L(x)$. Then since $\varepsilon(x) = xL'(x)/L(x)$,

$$\int_x^\infty h(t)\,\mathrm{d}t \leqslant L(x), \quad x \geqslant B. \tag{18}$$

Next since $\varepsilon(x) \to 0$ as $x \to \infty$, there is $x_0 \geqslant B$ such that $-\varepsilon(x) \leqslant 2/3$ for $x \geqslant x_0$. For such x,

$$3xh(x) = -3\varepsilon(x)L(x) \leqslant 2L(x),$$

hence

$$xh(x) \leqslant 2L(x) - 2xh(x) = 2\{L(x) + xL'(x)\}$$

so that

$$\int_{x_0}^x th(t)\mathrm{d}t \leqslant 2xL(x). \tag{19}$$

(16) and (17) then follow from (18) and (19), since $|\alpha_n| = \mathrm{O}(h(|n|))$ and $h(|n|)$ is nonincreasing.

Lemma 2. *Suppose that the condition* (15) *holds. Then there exists a constant K_3 such that*

$$\max_{|\lambda| \leqslant \lambda_0} |f(\lambda) - f(0)| \leqslant K_3 L(\lambda_0^{-1})$$

for all $0 < \lambda_0 \leqslant \pi$.

Proof. Let $j_\lambda = [|\lambda|^{-1}] + 1$. We have for $0 < |\lambda| \leqslant \lambda_0$,

$$\begin{aligned}
2\pi\sigma^{-2}|f(\lambda)-f(0)| &= \left|\left|\sum_{j=-\infty}^{\infty}\alpha_j e^{i\lambda j}\right|^2 - \left(\sum_{j=-\infty}^{\infty}\alpha_j\right)^2\right| \\
&= \left|\left(\sum_{j=-\infty}^{\infty}\alpha_j\cos\lambda j\right)^2 + \left(\sum_{j=-\infty}^{\infty}\alpha_j\sin\lambda j\right)^2 - \left(\sum_{j=-\infty}^{\infty}\alpha_j\right)^2\right| \\
&\leqslant \left|\left\{\sum_{j=-\infty}^{\infty}\alpha_j(1+\cos\lambda j)\right\}\left\{\sum_{j=-\infty}^{\infty}\alpha_j(1-\cos\lambda j)\right\}\right| \\
&\quad + \left(\sum_{j=-\infty}^{\infty}\alpha_j\sin\lambda j\right)^2 \\
&\leqslant 2\left(\sum_{j=-\infty}^{\infty}|\alpha_j|\right)\left\{\sum_{j=-\infty}^{\infty}|\alpha_j|(1-\cos\lambda j)\right. \\
&\quad \left. + \sum_{j=-\infty}^{\infty}|\alpha_j||\sin\lambda j|\right\} \\
&\leqslant 2\left(\sum_{j=-\infty}^{\infty}|\alpha_j|\right)\left(2|\lambda|\sum_{|j|\leqslant j_\lambda}|j||\alpha_j| + 3\sum_{|j|\geqslant j_\lambda}|\alpha_j|\right) \\
&\leqslant B_1(|\lambda|j_\lambda + 1)L(j_\lambda) \quad \text{(by Lemma 1)} \\
&\leqslant 6B_1 L(|\lambda|^{-1}) \leqslant 6B_1 L(\lambda_0^{-1}),
\end{aligned}$$

where B_1 is a constant not depending on λ. Here we used the elementary facts that $|\sin x| \leqslant 1$, $1 \pm \cos x \leqslant 2$, $|\sin x| \leqslant |x|$ and $1 - \cos x \leqslant |x|$.

For the process $\{X_k\}$ in (13), define sequences of random functions $\{\theta_n\}$ and $\{\eta_n\}$ by (9) and (10), respectively.

Theorem. *Suppose that* $E|\varepsilon_0|^p < \infty$ *for some* $p > 2$. *Let* $L(x) = (\log_2 x)\{(\log x)\cdots(\log_m x)^{1+\delta}\}^{-2/p}$ *for some* $\delta > 0$ *and* $m \geqslant 1$, *and suppose that*

$$|\alpha_n| = \mathrm{O}(|L'(|n|)|). \tag{20}$$

Then $\{\theta_n\}$ *converges weakly in* $C[0,1]$ *to* W. *Also,* $\{\eta_n\}$ *is relatively compact in* $C[0,1]$ *with probability one, and the set of its limit points coincides with* K.

Proof. Since $ET_n^2 = 2\pi f(0)n$,

$$\begin{aligned}
EU_n^2 &= E(S_n - T_n)^2 = ES_n^2 - 2ES_nT_n + ET_n^2 \\
&\leqslant |ES_n^2 - 2\pi f(0)n| + 2|ES_nT_n - 2\pi f(0)n|.
\end{aligned}$$

From the proof of Theorem 18.2.1 of Ibragimov and Linnik (1971),

$$\begin{aligned}
&|ES_n^2 - 2\pi f(0)\,n| \\
&\quad \leqslant 2\pi n \max_{|\lambda|\leqslant n^{-1/4}}|f(\lambda) - f(0)| + \mathrm{O}(n^{1/2}).
\end{aligned} \tag{21}$$

Next,

$$\begin{aligned} ES_nT_n &= \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_iY_j) = \sum_{|j|<n}(n-|j|)E(Y_0X_j) \\ &= \left(\sum_{j=-\infty}^{\infty}\alpha_j\right)\sigma^2 \sum_{|j|<n}(n-|j|)\alpha_j \\ &= 2\pi f(0)n - \sigma^2\left(\sum_{j=-\infty}^{\infty}\alpha_j\right)\left(n\sum_{|j|\geq n}\alpha_j + \sum_{|j|<n}|j|\alpha_j\right). \end{aligned} \tag{22}$$

Combining (21) and (22), we have

$$\begin{aligned} EU_n^2 &\leq 2\pi n \max_{|\lambda|\leq n^{-1/4}} |f(\lambda)-f(0)| \\ &\quad + 2\sigma^2\left|\sum_{j=-\infty}^{n}\alpha_j\right|\left(n\sum_{|j|\geq n}|\alpha_j| + \sum_{|j|<n}|j|\,|\alpha_j|\right) + \mathrm{O}(n^{1/2}). \end{aligned} \tag{23}$$

Since $L'(x) \sim -a(\log_2 x)\,(x\log x)^{-1}\,\{(\log x)\cdots(\log_m x)^{1+\delta}\}^{-2/p}$, where $a = 2/p$ or $(2/p)(1+\delta)$ according as $m \geq 2$ or $m = 1$, and $L(x^{1/4}) \sim 4^{2/p}L(x)$ as $x \to \infty$, from (23), Lemmas 1 and 2, there exists a constant K_4 such that

$$EU_n^2 \leq K_4\, nL(n), \quad n \geq 1. \tag{24}$$

We have

$$U_n = \sum_{j=-\infty}^{\infty} a_{nj}\varepsilon_j,$$

where $a_{nj} = \sum_{i=1-j}^{n-j}\alpha_i - \sum_{i=-\infty}^{\infty}\alpha_i$ if $1 \leq j \leq n$, and $a_{nj} = \sum_{i=1-j}^{n-j}\alpha_i$ otherwise. Then

$$\sum_{j=-\infty}^{\infty} a_{nj}^2 = \sigma^{-2}EU_n^2. \tag{25}$$

By Burkholder's and Hölder's inequalities, there exists a constant C_p depending only on p such that for all integers $k \leq h$,

$$\begin{aligned} E\left|\sum_{j=k}^{h} a_{nj}\varepsilon_j\right|^p &\leq C_pE\left(\sum_{j=k}^{h} a_{nj}^2\varepsilon_j^2\right)^{p/2} \\ &= C_pE\left(\sum_{j=k}^{h} |a_{nj}|^{(2p-4)/p}|a_{nj}|^{4/p}\varepsilon_j^2\right)^{p/2} \\ &\leq C_pE\left\{\left(\sum_{j=k}^{h} a_{nj}^2\right)^{p/2-1}\sum_{j=k}^{h} a_{nj}^2|\varepsilon_j|^p\right\} \\ &\leq C_p\left(\sum_{j=-\infty}^{\infty} a_{nj}^2\right)^{p/2} E|\varepsilon_0|^p. \end{aligned}$$

Hence by the Fatou lemma, (24) and (25), there exists a constant K_5 such that

$$E|U_n|^p \leq K_5(n\log_2 n)^{p/2}\,\{(\log n)\cdots(\log_m n)^{1+\delta}\}^{-1}, \quad n \geq 1,$$

and from Proposition 2 we obtain the desired result.

4. Remarks

(i) If the α_n are ultimately all positive, the condition (20) covers the cases $\alpha_{|n|} \sim Cn^{-\lambda}$ for some $\lambda > 1$, and $\alpha_{|n|} \sim Cn^{-1} (\log n)^{-1-\varepsilon}$ for some $\varepsilon \geqslant 2/p$.

(ii) The result of the Theorem remains valid even when T is not ergodic. In this case the α_j and σ^2, and hence $f(\lambda)$, and σ_n^2 can be defined as $\mathscr{I}$-measurable random variables, where $\mathscr{I} = \{A \in \mathscr{F} \mid T^{-1} A = A\}$. The proof is based on the decomposition of a nonergodic invariant measure into ergodic components (regular conditional probabilities induced by $\mathscr{I}$) (see Volný, 1987, 1993; Yokoyama and Volný, 1989).

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References

P. Billingsley, The Lindeberg–Lévy theorem for martingales, Proc. Amer. Math. Soc. 12 (1961) 788–792.

M.I. Gordin, The central limit theorem for stationary processes, Soviet Math. Dokl. 10 (1969) 1174–1176.

P. Hall and C. C. Heyde, Martingale Limit Theory and its Application (Academic Press, New York, 1980).

C.C. Heyde, On the central limit theorem and iterated logarithm law for stationary processes, Bull. Austral. Math. Soc. 12 (1975) 1–8.

C.C. Heyde and D.J. Scott, Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments, Ann. Probab. 1 (1973) 428–436.

I.A. Ibragimov, A central limit theorem for a class of dependent random variables, Theory Probab. Appl. 8 (1963) 83–89.

I.A. Ibragimov and Yu.V. Linnik, Independent and Stationary Sequences of Random Variables (Wolters-Noordhoff, Groningen, 1971).

T.L. Lai and W. Stout, Limit theorems for sums of dependent random variables, Z. Wahrsch. verw. Gebiete 51 (1980) 1–14.

E. Seneta, Regularly Varying Functions, Lecture Notes in Math. No. 508 (Springer, Berlin, 1976).

W.F. Stout, The Hartman–Wintner law of the iterated logarithm for martingales, Ann. Math. Statist. 41 (1970) 2158–2160.

D. Volný, On the invariance principle and functional law of iterated logarithm for nonergodic processes, Yokohama Math. J. 35 (1987) 137–141.

D. Volný, Approximating martingales and the central limit theorem for strictly stationary processes, Stochastic Process. Appl. 44 (1993) 41–74.

R. Yokoyama and D. Volný, On the central limit theorem and law of iterated logarithm for stationary linear processes, in: P. Mandl and M. Hušková, eds., Proc. Fourth Prague Symp. on Asymptotic Statistics (Charles Univ., Prague, 1989) pp. 535–538.